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O(3) nonlinear σ-model and pseudospherical surface

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Abstract. We relate the dual symmetry of the chiral field to the dual symmetry of the first and second forms on pseudospherical surfaces in asymptotic coordinates. Every given (up to conformal similarity) solution of chiral field on O(3)/O(2) sphere has been put into one-to-one correspondence with the Gauss image of a definite (up to homothetic transformations) pseudospherical surface. Thus we establish a gauge covariant formulation which unifies the chiral field and the differential forms on the pseudospherical surface. Then we get the explicit geometrical pictures and the covariant relations for Bäcklund transformations, Riccati equations and an infinite number of conserved currents in both cases. The SO(3) and SO(2, 1) σ-model are also related by dual symmetry.

1. Introduction

The well known completely integrable systems: chiral models and AKNS systems share many common features such as linear scattering equations, Bäcklund transformations (BTS), Riccati equations (RES), an infinite number of conservation laws, etc. (Pohlmeyer 1976, Luscher and Pohlmeyer 1978, Ogielski *et al* 1981, Eichenherr and Forger 1979, Ablowitz *et al* 1973, 1974, Zakharov and Shabat 1971, Crampin 1978, Chinea 1979, Sym and Corones 1979, Lund 1977, McIntosh 1981, Lamb 1976, 1977, Reiter 1980, Putko 1981). Further clarification of their intrinsic connections would be of benefit for a deeper understanding in both cases. There is already a lot of heuristic geometrical analysis in the literature for AKNS systems, e.g. Sasaki (1979) has shown that all the AKNS equations in $(1+1)$ dimensions (KdV, MKdV, sine-Gordon) describe pseudospherical surfaces (PSSS) (Sasaki 1979). Therefore it would be interesting to give the geometrical pictures of all the aforementioned features in the SO(3)/SO(2) chiral model by using PSSS.

Many authors have pointed out that the chiral model possesses a one parameter family of dual transformations (DTS), which leads to an infinite number of conservation laws. As one of the authors remarked in an earlier paper (Hou 1980), the equations of motion are in dual symmetry with the Gauss–Codazzi equations (GCES) (Hou 1983, Hou *et al* 1982). This dual property may be expressed as the dual matching between first and second fundamental forms on PSSS and induces the dual correspondence between SO(3) and SO(2, 1) symmetries. The former is the original symmetry of the SO(3)/SO(2) chiral field. It is the SO(3) symmetry of the Euclidean space embedding the PSS, i.e. it corresponds to the SO(3) rotations of the moving frames determined by the second fundamental forms on the PSS. The latter SO(2, 1) is the symmetry of the isometry group of these PSSS. This SO(2, 1) invariant metric is determined by the first fundamental forms. By constructing Killing vectors one may identify these two-dimensional metric spaces as the SO(2, 1)/SO(2) coset spaces embedding in the adjoint representation space of SO(2, 1), i.e. as a two-sheet hyperboloid in the 3-Minkowski space. Thus we get the SO(2, 1) σ-model naturally. This discrete dual symmetry may be generalised into DTS with continuous parameters. Then it is easy to see that the columns of the matrix representing these DTS of the chiral model are gauge equivalent to the scattering amplitudes of AKNS systems, and the parameter of DTS corresponds to the spectrum parameter.

This paper is organised as follows. In § 2 we discuss the DTS of the chiral field. In order to get the gauge covariant relations between the chiral model and PSS, all fundamental equations of the chiral model are expressed in gauge covariant forms. The explicit flat gauge and the explicitly reduced gauge have been introduced respectively. In § 3 it is shown that for any given solution of the chiral field, there is one and only one (up to homothetic transformations) PSS, such that the lightlike derivatives go along the asymptotic directions, while the chiral fields are induced by the normal image of this PSS. Meanwhile we have used the Euler–Lagrange equations (ELE) and GCES of the chiral field as the integrability conditions for this PSS. The dual symmetry between the SO(3) and SO(2, 1) σ-model is shown also. In § 4 we consider the geometrical picture of the BT of the chiral field and find its direct connection with the classical BT of PSS. In § 5 we show that the one-parameter family of BTS is obtained by dual transforming the Bianchi transformation. In § 6 we consider various RES, both in the chiral model and in the AKNS systems, find their covariant relations, and discuss the geometrical meaning of the solutions. In § 7 the infinitely many local conserved currents are derived as the gauge transformed form of the Nöether current of infinitesimal BT in the chiral model. In the appendix we consider the connection between the chiral model and the sine-Gordon equation and give the explicit expressions of the normal chiral field under reduced gauge in different coordinates on PSS.

2. Gauge covariant formulation and the DT of the chiral field

In this section we consider the O(3) σ-model field $N^a(x)$ $(a = 1, 2, 3)$ in $(1+1)$-dimensional space $\{x; \mu = 0, 1\}$. The dynamics are determined by the Lagrangian

$$L = \tfrac{1}{8}\mathrm{Tr}(\partial_\mu N(x)\partial^\mu N(x)) \tag{2.1}$$

with the constraint

$$N^2(x) = I, \tag{2.2}$$

where

$$N(x) = N^a(x)\sigma_a; \tag{2.3}$$

here σ_a are the Pauli matrices.

Thus, chiral fields $N^a(x)$ are harmonic maps from two-dimensional Minkowski space into symmetric space $S^2 \sim \mathrm{SO}(3)/\mathrm{SO}(2)$. The equation of motion (ELE) will be obtained from (2.1) and (2.2) by arbitrary variation of N in S^2:

$$[\partial_\mu \partial^\mu N(x), N(x)] = 0. \tag{2.4}$$

The Lagrangian has a global SO(3) symmetry generated by

$$T = \sigma_a T^a, \qquad T^a = \text{constant}.$$

Namely $\delta L = 0$ under

$$\delta N = [N, T].$$

Therefore we get the conserved Nöether current $J_\mu = \text{Tr}(K_\mu T)$, which satisfies

$$\partial_\mu \, \text{Tr}(K^\mu T) = 0, \tag{2.5}$$

where

$$K_\mu = -\tfrac{1}{2} N \partial_\mu N. \tag{2.6}$$

In (2.5) T may assume any constant value in su(2) algebra, so we have

$$\partial_\mu K^\mu = 0, \tag{2.7}$$

which is equivalent to (2.4).

From (2.2), (2.6) it is easy to see

$$\{K_\mu, N\} = 0, \tag{2.8}$$

i.e. $K_0(x)$, $K_1(x)$ as vectors in the adjoint space of SU(2)–E_3 are tangential on the sphere S^2 embedded in E_3. Usually it is desirable to express these vectors by using moving frames with the third axis along the normal direction $N(x)$, this implies that we introduce local transformations ('gauge' transformations) as follows:

$$K^\mu(x) \to k^\mu(x) = g^{-1}(x) K^\mu(x) g(x) \tag{2.9a}$$

where $g(x)$ satisfy

$$g^{-1}(x) N(x) g(x) = n = \sigma_3. \tag{2.9b}$$

To get a unified SU(2) gauge covariant formulation for both $K_\mu(x)$ and $k_\mu(x)$ we would like to discuss the more general case i.e.

$$K_\mu(x) \to K_\mu^{(S)}(x) = S^{-1}(x) K_\mu(x) S(x) \tag{2.10a}$$

$$N(x) \to N^{(S)}(x) = S^{-1}(x) N(x) S(x) \tag{2.10b}$$

where $S(x)$ are arbitrary SU(2) matrices, i.e. in arbitrary moving frames with the third axis not necessarily coinciding with the normal direction. Now instead of the equation (2.7), we have

$$\begin{aligned} \partial^\mu K_\mu^{(S)}(x) &= -S^{-1}\partial^\mu S K_\mu^{(S)}(x) + K_\mu^{(S)}(x) S^{-1}\partial^\mu S \\ &\equiv -[A^{(S)\mu}(x), K_\mu^{(S)}(x)] \end{aligned}$$

or

$$D^{(S)\mu} K_\mu^{(S)} = 0, \qquad D^{(S)\mu} \equiv \partial^\mu + [A^{(S)\mu}]. \tag{2.11}$$

Thus by introducing formally a non-dynamical pure gauge

$$A^{(S)\mu}(x) \equiv S^{-1}(x) \partial^\mu S(x), \tag{2.12}$$

we get a gauge covariant version of the equation of motion (2.7). From (2.10), (2.12) it is easy to show that the gauge transformations between two different gauges $S'(x)$,

$S''(x)$ are

$$K''_\mu = S^{-1}K'_\mu S, \qquad N'' = S^{-1}N'S, \qquad A''_\mu = S^{-1}A'_\mu S + S^{-1}\partial_\mu S \tag{2.13a, b, c}$$

where $S = S'^{-1}S''$. Besides the gauge covariance of $K_\mu^{(S)}$ and the gauge covariant relation

$$\mathrm{Tr}(K_\mu^{(S)}N^{(S)}) = 0 \tag{2.14}$$

we may express $K_\mu^{(S)}(x)$ gauge covariantly in terms of $A_\mu^{(S)}(x)$ and $N^{(S)}(x)$

$$\begin{aligned} K_\mu^{(S)} &= S^{-1}K_\mu S = -\tfrac{1}{2}S^{-1}N\partial_\mu NS \\ &= -\tfrac{1}{2}N^{(S)}D_\mu^{(S)}N^{(S)}. \end{aligned} \tag{2.15}$$

Hence if we subdivide $A_\mu^{(S)}$ by introducing

$$H_\mu^{(S)} \equiv A_\mu^{(S)} - K_\mu^{(S)} \tag{2.16}$$

then from (2.13), (2.16) we see that $H_\mu^{(S)}$ changes as an SU(2) connection (gauge potential) under the general SU(2) gauge transformation (2.13)

$$H''_\mu = S^{-1}H'_\mu S + S^{-1}\partial_\mu S. \tag{2.17}$$

The crucial point for introducing $H_\mu^{(S)}$ lies in the fact that $H_\mu^{(S)}$ possesses an important property

$$\mathscr{D}_\mu^{(S)}N^{(S)} \equiv \partial_\mu N^{(S)} + [H_\mu^{(S)}, N^{(S)}] = D_\mu^{(S)}N^{(S)} - [K_\mu^{(S)}, N^{(S)}] = 0, \tag{2.18}$$

i.e. $H_\mu^{(S)}$ can be reduced to a U(1) connection since there exists a section $N(x)$ in coset bundle $SO(3)/SO(2) \sim S^2$ which is invariant under the parallel translation with respect to the connection $H_\mu^{(S)}$. Hence the SU(2) covariant equation of motion (2.11) further simplifies as

$$\mathscr{D}_\mu^{(S)}K^{(S)\mu} \equiv \partial^\mu K_\mu^{(S)} + [H^{(S)\mu}, K_\mu^{(S)}] = 0 \tag{2.19}$$

with an essentially U(1) connection $H_\mu^{(S)}$. We may express $H_\mu^{(S)}$ also in terms of $N^{(S)}$ and $A_\mu^{(S)}$

$$\begin{aligned} H_\mu^{(S)} &= -K_\mu^{(S)} + A_\mu^{(S)} = \tfrac{1}{2}N^{(S)}D_\mu^{(S)}N^{(S)} + A_\mu^{(S)} \\ &= \tfrac{1}{2}N^{(S)}\partial_\mu N^{(S)} + \tfrac{1}{2}\mathrm{Tr}(A_\mu^{(S)}N^{(S)})N^{(S)}. \end{aligned} \tag{2.20}$$

Thus for given connection $A_\mu^{(S)}$ and $N^{(S)}$ we can always divide $A_\mu^{(S)}$ into a connection $H_\mu^{(S)}$ and a covariant $K_\mu^{(S)}$ as (2.15) and (2.20) with the covariant property (2.14) and (2.18), such that $H_\mu^{(S)}$ can be locally reducible to U(1), while in the explicitly reduced gauge, $k_\mu(x)$ (we denote the fields in reduced gauge by corresponding small letters) becomes explicitly horizontal.

In the special cases of $S = 1$, we have

$$A_\mu^{(1)} = 0, \qquad K_\mu^{(1)} \equiv K_\mu = -\tfrac{1}{2}N\partial_\mu N, \qquad H_\mu^{(1)} = \tfrac{1}{2}N\partial_\mu N \tag{2.21a, b, c}$$

(more properly we should use the notation $A_\mu^{(1)}$, $K_\mu^{(1)}$, $H_\mu^{(1)}$, $N^{(1)}$ for A_μ, K_μ, H_μ, N respectively, but in order not to be burdened with superscripts we omit the superscript (1)).

This is called the explicit flat gauge. In this gauge we see explicitly the existence of 'absolute parallel frames'—generators of symmetry—T_i($T_i^a = \delta_i^a$, $i = 1, 2, 3$) over all points in space x, underlying a (1+1)-dimensional Minkowski base. Now we gauge transform $H_\mu(x)$ to the explicitly reduced gauge, such that in gauge transformed frame,

$N(x)$ is everywhere assumed to have constant n independent of x

$$g^{-1}(x)N(x)g(x)=n\equiv\sigma_3, \tag{2.22}$$

meanwhile there still remains a U(1) gauge freedom

$$g(x)\to g'(x)=g(x)h(x)$$

where $h(x)$ satisfies

$$h(x)nh^{-1}(x)=n.$$

Substituting $S=g(x)$ in (2.12), (2.20), (2.15), we obtain the corresponding a_μ, h_μ, k_μ in the explicitly reduced gauge:

$$\begin{aligned} a_\mu(x)&=h_\mu(x)+k_\mu(x)=g^{-1}(x)\partial_\mu g(x),\\ h_\mu(x)&=\tfrac{1}{2}\{g^{-1}\partial_\mu g,\, n\}n,\\ k_\mu(x)&=\tfrac{1}{2}[g^{-1}\partial_\mu g,\, n]n, \end{aligned} \tag{2.23}$$

while (2.8), (2.18) become respectively

$$\{k_\mu, n\}=0, \qquad [h_\mu, n]=0. \tag{2.24}$$

Namely with respect to the involutive operator n, the pure gauge $a_\mu(x)$ decomposes naturally into two parts, horizontal $k_\mu(x)$ and vertical $h_\mu(x)$.

By means of

$$\mathrm{Tr}(N^{(S)}(\mathscr{D}_\mu^{(S)}K_\nu^{(S)}-\mathscr{D}_\nu^{(S)}K_\mu^{(S)}))=0$$

we can express the zero curvature condition of the gauge potential $A_\mu^{(S)}$ as the GCEs

$$\tfrac{1}{2}\mathrm{Tr}(F_{\mu\nu}^{(S)}N^{(S)})N^{(S)}=\partial_\mu H_\nu^{(S)}-\partial_\nu H_\mu^{(S)}+[H_\mu^{(S)}, H_\nu^{(S)}]+[K_\mu^{(S)}, K_\nu^{(S)}]=0, \tag{2.25a}$$

$$F_{\mu\nu}^{(S)}-\tfrac{1}{2}\mathrm{Tr}(F_{\mu\nu}^{(S)}N^{(S)})N^{(S)}=\mathscr{D}_\mu^{(S)}K_\nu^{(S)}-\mathscr{D}_\nu^{(S)}K_\mu^{(S)}=0, \tag{2.25b}$$

where

$$F_{\mu\nu}^{(S)}=\partial_\mu A_\nu^{(S)}-\partial_\nu A_\mu^{(S)}+[A_\mu^{(S)}, A_\nu^{(S)}].$$

In the explicitly reduced gauge these equations become

$$\partial_\mu h_\nu-\partial_\nu h_\mu+[h_\mu, h_\nu]=-[k_\mu, k_\nu] \tag{2.26}$$

$$=[{}^*k_\mu, {}^*k_\nu], \tag{2.27}$$

$$\mathscr{D}_\mu{}^*k^\mu\equiv\partial_\mu{}^*k^\mu+[h_\mu, {}^*k^\mu]=0, \tag{2.28}$$

where

$${}^*k_\mu=\varepsilon_{\mu\nu}k^\nu \qquad (\varepsilon_{01}=-\varepsilon_{10}=1). \tag{2.29}$$

These GCEs are gauge covariant. They are trivial identities under the original flat gauge. Under the reduced gauge the equation of motion (2.19) becomes

$$\mathscr{D}_\mu k^\mu\equiv\partial_\mu k^\mu+[h_\mu, k^\mu]=0. \tag{2.30}$$

The GCEs (2.26)–(2.28) and ELE (2.30) are the gauge covariant fundamental equations. They are mutually dual under $k_\mu\leftrightarrow \mathrm{i}{}^*k_\mu$. From equations (2.28) and (2.30) we notice that k_μ and ${}^*k_\mu$ satisfy the same equation, so for the solution of ELE (2.30) of the chiral field, we can use the real linear combinations of k_μ and $*k_\mu$ as follows:

$$\tilde{k}_\mu\langle\gamma\rangle = \tilde{k}_\mu(x;\gamma) = \cosh\varphi k_\mu(x) + \sinh\varphi {}^*k_\mu(x), \tag{2.31}$$

$$*\tilde{k}_\mu\langle\gamma\rangle = *\tilde{k}_\mu(x;\gamma) = \cosh\varphi {}^*k_\mu(x) + \sinh\varphi k_\mu(x),$$

where

$$\cosh\varphi \equiv \tfrac{1}{2}(\gamma+\gamma^{-1}), \qquad \sinh\varphi \equiv \tfrac{1}{2}(\gamma-\gamma^{-1}). \tag{2.32}$$

It is easy to see that h_μ, $\tilde{k}_\mu\langle\gamma\rangle$ satisfy similar fundamental equations (2.26)-(2.30) to h_μ, k_μ respectively. That is $h_\mu(x)+\tilde{k}_\mu(x;\gamma)$ is a pure gauge also. Hence, there exists some $u(x;\gamma)\equiv u\langle\gamma\rangle$ such that

$$h_\mu(x)+\tilde{k}_\mu(x;\gamma) = u(x;\gamma)\partial_\mu u^{-1}(x;\gamma) = -\partial_\mu u(x;\gamma)u^{-1}(x;\gamma). \tag{2.33}$$

In case of $\gamma=1$, this equation becomes (2.23) with

$$u(x;1) = g^{-1}(x). \tag{2.34}$$

Since h_μ and $\tilde{k}_\mu\langle\gamma\rangle$ satisfy similar equations to h_μ and k_μ, it is easy to show that they really give a chiral field $N(x;\gamma)\equiv N\langle\gamma\rangle$ after gauge transformation to the corresponding flat gauge by choosing $S=u^{-1}\langle\gamma\rangle$ i.e. let;

$$H_\mu\langle\gamma\rangle = u^{-1}\langle\gamma\rangle h_\mu u\langle\gamma\rangle + u^{-1}\langle\gamma\rangle\partial_\mu u\langle\gamma\rangle, \tag{2.35}$$

$$K_\mu\langle\gamma\rangle = u^{-1}\langle\gamma\rangle = \tilde{k}_\mu\langle\gamma\rangle u\langle\gamma\rangle,$$

$$N\langle\gamma\rangle = u^{-1}\langle\gamma\rangle n u\langle\gamma\rangle.$$

Then from equations (2.33), (2.35) we obtain

$$K_\mu\langle\gamma\rangle = -H_\mu\langle\gamma\rangle = -\tfrac{1}{2}N\langle\gamma\rangle\partial_\mu N\langle\gamma\rangle; \tag{2.36}$$

from (2.30) and its gauge covariance we obtain

$$\partial_\mu K^\mu\langle\gamma\rangle + [H_\mu\langle\gamma\rangle, K^\mu\langle\gamma\rangle] = 0. \tag{2.37}$$

Thus we see that $N\langle\gamma\rangle$ is a new solution of the ELE (2.4). This solution may be related directly to the original $N(x)$

$$N\langle\gamma\rangle = U^{-1}\langle\gamma\rangle N(x) U\langle\gamma\rangle, \tag{2.38}$$

where

$$U\langle\gamma\rangle = g(x)u\langle\gamma\rangle. \tag{2.39}$$

Here we have used (2.22), (2.35) and chosen (2.34) to get

$$U\langle 1\rangle = I \tag{2.40}$$

where I is the 2×2 unit matrix. From (2.39), (2.33), (2.23) the operators $U\langle\gamma\rangle$ satisfy

$$\partial_\mu U\langle\gamma\rangle = -(H_\mu + \tilde{K}_\mu\langle\gamma\rangle)U\langle\gamma\rangle, \tag{2.41}$$

where

$$\tilde{K}_\mu\langle\gamma\rangle = g\tilde{k}_\mu\langle\gamma\rangle g^{-1} = \cosh K_\mu + \sinh\varphi {}^*K_\mu \tag{2.42}$$

$$= U\langle\gamma\rangle K_\mu\langle\gamma\rangle U^{-1}\langle\gamma\rangle. \tag{2.43}$$

Thus the dual transforming operator $U\langle\gamma\rangle$ in (2.41) (Pohlmeyer 1976) may be understood to be gauge equivalent to the expression (2.33), conversely, the integrability conditions of (2.41) or (2.33) respectively imply the equations of motion and GCEs in flat or reduced gauge in each turn.

3. From the chiral field to PSS

In this section we shall show that for any given solution of the chiral field $N(x)$, we may find one and only one PSS $X^i(x)$ $(i = 1, 2, 3)$ with curvature -1 which is embedding in the same three-dimensional Euclidean space (the adjoint representation space of SU(2)) as the chiral sphere, such that the lightlike direction $(\xi = t + x,\ \ \eta = t - x)$

$$\xi = \text{constant} \qquad \text{or} \qquad \eta = \text{constant}$$

on the base space x have been mapped onto the PSS X as the asymptotic curves, while the spherical representation of the normals of X induce the given chiral solutions on S^2. Now we construct the second and first fundamental forms ω_{ij}, ω_i of the required PSS in terms of h_μ and k_μ, which represent the chiral field in reduced gauge. Then, using the fundamental equations of chiral field (2.26)-(2.30), we can prove the integrability of the structural equations of the required PSS. Thus the surface with above mentioned ω_{ij} and ω_i does actually exist, meanwhile it happens that the curvature of the constructed surface is really -1.

Let $\boldsymbol{e}_i(x)$ $(i = 1, 2, 3)$ be the moving frame fields along the required PSS X, here $\boldsymbol{e}_3(x)$ has been chosen as the normal direction, and we let

$$\boldsymbol{e}_3(x) = \boldsymbol{N}(x)$$

to ensure that the normal image of the PSS describes the given chiral field $N(x)$. We have the Gauss-Weingarten formula

$$\mathrm{d}\boldsymbol{e}_i(x) = \omega_{ij}(x)\boldsymbol{e}_j(x), \tag{3.1}$$

where $\omega_{ij}(x)$ are the second fundamental forms of X (pulled back to the base space x).

For consistency with (2.22), we choose

$$g^{-1}(x)e_i^a(x)\sigma_a g(x) = \sigma_i. \tag{3.2}$$

Differentiating the above equations and using (3.1) it is easily verified that

$$[g^{-1}\,\mathrm{d}g, \sigma_i] = \omega_{ij}\sigma_j, \tag{3.3}$$

so we get

$$g^{-1}\,\mathrm{d}g = -\tfrac{1}{4}\mathrm{i}\varepsilon^{abc}\omega_{bc}\sigma_a$$

$$= -\tfrac{1}{2}\mathrm{i}\begin{pmatrix} \omega_{12} & \omega_{23} + \mathrm{i}\omega_{13} \\ \omega_{23} - \mathrm{i}\omega_{13} & -\omega_{12} \end{pmatrix} \equiv -\Omega^{\mathrm{II}}. \tag{3.4}$$

If this result is compared with (2.23), we then obtain the relations between the chiral field h_μ, k_μ and the second fundamental forms:

$$-(h_\mu\,\mathrm{d}x^\mu + k_\mu\,\mathrm{d}x^\mu) = \tfrac{1}{2}\mathrm{i}\begin{pmatrix} \omega_{12} & \omega_{23} + \mathrm{i}\omega_{13} \\ \omega_{23} - \mathrm{i}\omega_{13} & -\omega_{12} \end{pmatrix} \equiv \Omega^{\mathrm{II}}, \tag{3.5}$$

i.e.

$$h_\mu\,\mathrm{d}x^\mu = -\tfrac{1}{2}\mathrm{i}\omega_{12}\sigma_3, \tag{3.6}$$

$$k_\mu\,\mathrm{d}x^\mu = -\tfrac{1}{2}\mathrm{i}(\omega_{23}\sigma_1 - \omega_{13}\sigma_2). \tag{3.7}$$

Then integrability conditions $\mathrm{d}\mathrm{d}\boldsymbol{e}_i = 0$ may be written as

$$\mathrm{d}\Omega^{\mathrm{II}} = \Omega^{\mathrm{II}} \wedge \Omega^{\mathrm{II}}, \tag{3.8}$$

or

$$\mathrm{d}\omega_{12} = -\omega_{13} \wedge \omega_{23}, \qquad \mathrm{d}\omega_{13} = \omega_{12} \wedge \omega_{23}, \qquad \mathrm{d}\omega_{23} = -\omega_{12} \wedge \omega_{13}. \tag{3.9}$$

By means of (3.5) the above equations in terms of h_μ, k_μ are the GCEs of the original chiral field (2.26), (2.30). We see the algebra of Ω^{II} is su(2), which is the symmetry of the original chiral field (the symmetry of its Lagrangian). Obviously, equation (3.8) is the integrability condition of (2.23) or (2.33) with $\gamma = 1$, which may be written as

$$\mathrm{d}u(x; 1) = \Omega^{\mathrm{II}} u(x; 1), \tag{3.10}$$

here

$$u(x; 1) = g^{-1}(x).$$

Now we turn to construct the first fundamental forms, which satisfy

$$\mathrm{d}\boldsymbol{X} = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2. \tag{3.11}$$

Since the discrete dual symmetry of the chiral fields $\mathrm{i}^* k_\mu$, h_μ satisfy the similar equations as k_μ, h_μ. Define

$$\omega_1(x) = {}^*\omega_{23}(x), \qquad \omega_2(x) = -{}^*\omega_{13}(x), \tag{3.12}$$

where * is the Hodge star of the differential forms

$$^*\mathrm{d}t = \mathrm{d}x, \qquad ^*\mathrm{d}x = \mathrm{d}t \qquad (\text{i.e. } ^*\mathrm{d}x_\mu = \varepsilon_{\mu\nu}\, \mathrm{d}x^\nu),$$

so

$$^*\mathrm{d}\xi = \mathrm{d}\xi, \qquad ^*\mathrm{d}\eta = -\mathrm{d}\eta \qquad (^{**}\ \text{identity transformation}). \tag{3.13}$$

By using (3.7), (3.12), (2.29) we can express ω_i in terms of $^*k_\mu$

$$\begin{aligned} \tfrac{1}{2}(\omega_1\sigma_1 + \omega_2\sigma_2) &= \tfrac{1}{2}(^*\omega_{23}\sigma_1 - {}^*\omega_{13}\sigma_2) = \mathrm{i}^*(k_\mu\, \mathrm{d}x^\mu) \\ &= \mathrm{i}^* k_\mu\, \mathrm{d}x^\mu. \end{aligned} \tag{3.14}$$

So the integrability conditions are respectively as follows

(1) $$\omega_a \wedge \omega_{a3} = 0. \tag{3.15}$$

By means of (3.7), (3.14) the above equation becomes

$$\mathrm{Tr}([N, K_\mu]^* K_\nu)\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{Tr}(\partial_\mu N \varepsilon_{\nu\lambda} N \partial_\lambda N)\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = 0,$$

so (3.15) really is satisfied.

(2) $$\mathrm{d}\omega_1 = \omega_{12} \wedge \omega_2, \qquad \mathrm{d}\omega_2 = \omega_1 \wedge \omega_{12}. \tag{3.16}$$

By virtue of (3.6), (3.14) above equations are identical to the equations of motion (or equivalently the dual transformed Codazzi equations), so they are satisfied also.

In short, the structural equations with ansatz (3.6), (3.14) are integrable, i.e. the surface with such fundamental forms exist. Now we consider the property of this surface. From (3.9), (3.12) it is easy to show that

$$\mathrm{d}\omega_{12} = -\omega_{13} \wedge \omega_{23} = {}^*\omega_{13} \wedge {}^*\omega_{23} = -\omega_2 \wedge \omega_1. \tag{3.17}$$

This implies that the curvature of surface X is -1 everywhere, i.e. it is a PSS.

By virtue of (3.1), (3.11), (3.7) and (3.14)

$$\mathrm{d}\boldsymbol{e}_3 \cdot \mathrm{d}\boldsymbol{X} = \tfrac{1}{2}\mathrm{i}\, \mathrm{Tr}(N[K_\xi, K_\eta])\, \mathrm{d}\xi\, \mathrm{d}\eta, \tag{3.18}$$

so $\mathrm{d}\xi$ and $\mathrm{d}\eta$ are the asymptotic directions. Notice the dual symmetry (3.12) of fundamental forms on PSS in the asymptotic system of coordinates, let

$$h_\mu\, \mathrm{d}x^\mu + \mathrm{i}^*k_\mu\, \mathrm{d}x^\mu = -\tfrac{1}{2}\begin{pmatrix} \mathrm{i}\omega_{12} & \omega_1 - \mathrm{i}\omega_2 \\ \omega_1 + \mathrm{i}\omega_2 & -\mathrm{i}\omega_{12} \end{pmatrix} \equiv -\Omega^{\mathrm{I}} \tag{3.19}$$

then the equations (3.16), (3.17) can be collected together in the similar way as equation (3.8)

$$\mathrm{d}\Omega^{\mathrm{I}} = \Omega^{\mathrm{I}} \wedge \Omega^{\mathrm{I}}. \tag{3.20}$$

They are the ELE and dual transformed Gauss equation of the original chiral field (2.27), (2.30). The algebra of Ω^{I} is su(1, 1), this is the symmetry of isometry group of the surface with negative constant curvature. Similarly as (3.10) we can write Ω^{I} as the Maurer-Cartan form of group SU(1, 1)

$$\mathrm{d}v(x) = \Omega^{\mathrm{I}} v(x) \tag{3.21}$$

where $v(x) \in \mathrm{SU}(1, 1)$. The integrability condition of this equation is the Maurer–Cartan equation (3.20).

Above we apply only the discrete dual symmetry (Hodge dual of K_μ) of the fundamental equations (2.26)–(2.30). Below we further apply the symmetry of continuous DT. Then from a given solution of chiral field $N(x)$, we can construct a family of different PSSS $\boldsymbol{X}\langle\gamma\rangle$ (γ is a real parameter). For this purpose we introduce a family of $\omega_{ij}(x;\gamma) \equiv \tilde{\omega}_{ij}$ as follows:

$$\begin{aligned} &\tilde{\omega}_{12} = \omega_{12}, \qquad \tilde{\omega}_{23} = \cosh \varphi\omega_{23} + \sinh \varphi\omega_1, \\ &\tilde{\omega}_{13} = \cosh \varphi\omega_{13} - \sinh \varphi\omega_2, \qquad \tilde{\omega}_1 = \cosh \varphi\omega_1 + \sinh \varphi\omega_{23}, \\ &\tilde{\omega}_2 = \cosh \varphi\omega_2 - \sinh \varphi\omega_{13}. \end{aligned} \tag{3.22}$$

We can prove as before that the structural equations of the system $\tilde{\omega}_{ij}$, $\tilde{\omega}_i$ are integrable, the surface $\boldsymbol{X}\langle\gamma\rangle$ exists, and its curvature is -1. Now by means of the Lie transformations of the original PSS, we obtain a family of new PSSS. Meanwhile the equations (3.5), (3.10), (3.19) and (3.21) may be generalised as follows:

$$\mathrm{d}u\langle\gamma\rangle = \Omega^{\mathrm{II}}\langle\gamma\rangle u\langle\gamma\rangle, \tag{3.23}$$

where

$$\begin{aligned} \Omega^{\mathrm{II}}\langle\gamma\rangle &\equiv \tfrac{1}{2}\mathrm{i}\begin{pmatrix} \omega_{12} & \tilde{\omega}_{23} + \mathrm{i}\tilde{\omega}_{13} \\ \tilde{\omega}_{23} - \mathrm{i}\tilde{\omega}_{13} & -\omega_{12} \end{pmatrix} \\ &= -(h_\mu\, \mathrm{d}x^\mu + \tilde{k}_\mu\, \mathrm{d}x^\mu), \end{aligned} \tag{3.24}$$

and

$$\mathrm{d}v\langle\gamma\rangle = \Omega^{\mathrm{I}}\langle\gamma\rangle v\langle\gamma\rangle, \tag{3.25}$$

where

$$\begin{aligned} \Omega^{\mathrm{I}}\langle\gamma\rangle &\equiv \tfrac{1}{2}\begin{pmatrix} \mathrm{i}\omega_{12} & \tilde{\omega}_1 - \mathrm{i}\tilde{\omega}_2 \\ \tilde{\omega}_1 + \mathrm{i}\tilde{\omega}_2 & -\mathrm{i}\omega_{12} \end{pmatrix} \\ &= -(h_\mu\, \mathrm{d}x^\mu + \mathrm{i}^*\tilde{k}_\mu\, \mathrm{d}x^\mu). \end{aligned} \tag{3.26}$$

By using the fundamental equations of the chiral field it is easy to see that both above systems of equations are integrable, i.e.

$$d\Omega^{II}\langle\gamma\rangle = \Omega^{II}\langle\gamma\rangle \wedge \Omega^{II}\langle\gamma\rangle, \qquad d\Omega^{I}\langle\gamma\rangle = \Omega^{I}\langle\gamma\rangle \wedge \Omega^{I}\langle\gamma\rangle. \tag{3.27}$$

The former (3.23) is the su(2) linear scattering equation in Lund and Regge (1976), and the latter (3.25) is the su(1, 1) linear scattering equation in Sasaki (1979).

From (2.31) we get

$$\tilde{k}_\mu\langle i\gamma\rangle = i^*\tilde{k}\langle\gamma\rangle, \tag{3.28}$$

so that

$$\Omega^{II}\langle i\gamma\rangle = \Omega^{I}\langle\gamma\rangle, \tag{3.29}$$

hence

$$u\langle i\gamma\rangle = v\langle\gamma\rangle.$$

This shows that when the parameter γ is an imaginary number, the algebra of symmetry changes from su(2) to su(1, 1). In fact, if we define the extension of the solution of the chiral model

$$N\langle i\gamma\rangle = M\langle\gamma\rangle, \tag{3.30}$$

$$\begin{aligned} M\langle\gamma\rangle &= U^{-1}\langle i\gamma\rangle N U\langle i\gamma\rangle = u^{-1}\langle i\gamma\rangle n u\langle i\gamma\rangle \\ &= v^{-1}\langle\gamma\rangle n v\langle\gamma\rangle = v^{-1}\langle\gamma\rangle\sigma_3 v\langle\gamma\rangle. \end{aligned} \tag{3.31}$$

It is easy to see that M is a unit vector in O(2, 1) adjoint space (three-dimensional Minkowski space),

$$M^a M_a = 1, \qquad M^a = \tfrac{1}{2}\mathrm{Tr}(M\rho^a), \tag{3.32}$$

where ρ^a is the generators of su(1, 1)

$$\rho_3 = \sigma_3, \qquad \rho_1 = i\sigma_1, \qquad \rho_2 = i\sigma_2, \tag{3.33}$$

$$g_{33} = -g_{11} = -g_{22} = 1. \tag{3.34}$$

Now M is a chiral field along SO(2, 1)/SO(2), which appears as the surface of a hyperboloid in three-dimensional Minkowski space. The metric on hyperboloid may be given as

$$\begin{aligned} \tfrac{1}{2}\mathrm{Tr}(dM\langle\gamma\rangle)^2 &= \tfrac{1}{2}\mathrm{Tr}(dN\langle i\gamma\rangle)^2 = -2\,\mathrm{Tr}(K_\mu\langle i\gamma\rangle K_\nu\langle i\gamma\rangle\, dx^\mu\, dx^\nu) \\ &= 2\,\mathrm{Tr}({}^*K_\mu\langle\gamma\rangle {}^*K_\nu\langle\gamma\rangle\, dx^\mu\, dx^\nu) = -(d\boldsymbol{X}\langle\gamma\rangle)^2. \end{aligned} \tag{3.35}$$

It is interesting to point out that a surface with the same intrinsic metric (3.35) may be embedded in two ways, either as a PSS $\boldsymbol{X}\langle\gamma\rangle$ in Euclidean space or as a hyperboloid $M\langle\gamma\rangle$ in Minkowski space.

4. BT of chiral field and its geometrical picture

The BT equation of chiral field has been given as in an earlier paper (Hou 1980)

$$N\partial_\mu N - N'\partial_\mu N' = \varepsilon_{\mu\nu}\partial^\nu(NN'). \tag{4.1}$$

It is easy to see that if $N(x)$ is a solution of the ELE (2.4), then $N'(x)$ in the above equation is also a solution and *vice versa*.

In § 3 we show that the chiral field $N(x)$ may be described by the Gauss image of the PSS. As is well known, there exist BTS between the PSSS. In this section we shall show that the equation (4.1) is equivalent to the classical BT between the PSSS, thus give the equation (4.1) some explicit geometric pictures.

Let

$$N(x)=N^a(x)\sigma_a=\boldsymbol{e}_3(x)\cdot\boldsymbol{\sigma}, \qquad N'(x)=N'^a(x)\sigma_a=\boldsymbol{e}'_3(x)\cdot\boldsymbol{\sigma}, \tag{4.2}$$

and choose the common tangents of corresponding PSSS X and X' as

$$\boldsymbol{e}_1(x)=\boldsymbol{e}'_1(x)=\boldsymbol{e}_3(x)\boldsymbol{e}'_3(x)/|\boldsymbol{e}_3(x)\boldsymbol{e}'_3(x)|. \tag{4.3}$$

Express the matrix NN' in terms of $\boldsymbol{e}_1$,

$$B\equiv NN'=\cos\theta I+\mathrm{i}\sin\theta(\boldsymbol{e}_1\cdot\boldsymbol{\sigma}), \tag{4.4}$$

where θ is the angle between $\boldsymbol{N}(x)$ and $\boldsymbol{N}'(x)$. By virtue of equation (4.1) and $N^2=N'^2=I$ we get

$$\partial_\mu(B+B^+)=0 \tag{4.5}$$

so

$$B+B^+=2\boldsymbol{e}_3(x)\cdot\boldsymbol{e}'_3(x)=2\cos\theta I=\text{constant}. \tag{4.6}$$

Hence θ is a constant not relying on t and x. From (4.3)

$$\begin{aligned}\boldsymbol{e}'_2(x)&=\cos\theta\boldsymbol{e}_2(x)+\sin\theta\boldsymbol{e}_3(x),\\ \boldsymbol{e}'_3(x)&=\cos\theta\boldsymbol{e}_3(x)-\sin\theta\boldsymbol{e}_2(x).\end{aligned} \tag{4.7}$$

The Gauss–Weingarten formula is written as

$$\mathrm{d}\boldsymbol{e}_i=\omega_{ij}\boldsymbol{e}_j, \qquad \mathrm{d}\boldsymbol{e}'_i=\omega'_{ij}\boldsymbol{e}'_j. \tag{4.8}$$

Since θ is a constant, we get

$$\begin{aligned}\omega'_{23}&=\omega_{23},\\ \omega'_{13}&=\cos\theta\omega_{13}-\sin\theta\omega_{12},\\ \omega'_{12}&=\cos\theta\omega_{12}+\sin\theta\omega_{13}.\end{aligned} \tag{4.9}$$

From (4.2), (4.8) we get

$$\begin{aligned}N\,\mathrm{d}N&=\mathrm{i}[-\omega_{13}(\boldsymbol{e}_2\cdot\boldsymbol{\sigma})+\omega_{23}(\boldsymbol{e}_1\cdot\boldsymbol{\sigma})],\\ N'\,\mathrm{d}N'&=\mathrm{i}[-\omega'_{13}(\boldsymbol{e}'_2\cdot\boldsymbol{\sigma})+\omega'_{23}(\boldsymbol{e}'_1\cdot\boldsymbol{\sigma})].\end{aligned} \tag{4.10}$$

By using (4.9), (4.10) the left-hand side of (4.1) becomes

$$N\,\mathrm{d}N-N'\,\mathrm{d}N'=-\mathrm{i}\sin\theta[(\sin\theta\omega_{13}+\cos\theta\omega_{12})\boldsymbol{e}_2\cdot\boldsymbol{\sigma}+(\sin\theta\omega_{12}-\cos\theta\omega_{13})\boldsymbol{e}_3\cdot\boldsymbol{\sigma}],$$

the right-hand side of (4.1) yields

$$\varepsilon_{\mu\nu}\partial^\mu B\,\mathrm{d}x^\mu=-{}^*(\mathrm{d}B)=-\mathrm{i}\sin\theta({}^*\omega_{12}\boldsymbol{e}_2\cdot\boldsymbol{\sigma}+{}^*\omega_{13}\boldsymbol{e}_3\cdot\boldsymbol{\sigma}).$$

By comparing corresponding terms we get

$${}^*\omega_{12}=\sin\theta\omega_{13}+\cos\theta\omega_{12}, \tag{4.11}$$

$${}^*\omega_{13}=\sin\theta\omega_{12}-\cos\theta\omega_{13}. \tag{4.12}$$

Substituting (3.12) in (4.12), we get

$$\omega_2 = \cos\theta\omega_{13} - \sin\theta\omega_{12}, \tag{4.13}$$

which is the classical BT equation in the coordinate systems with the common tangent as base $\boldsymbol{e}_1$ (Chern and Terng 1980).

After taking the Hodge star of (4.11), we get

$$\omega_{12} = \sin\theta^*\omega_{13} + \cos\theta^*\omega_{12},$$

by substituting this in (4.11) we may eliminate $^*\omega_{12}$ to obtain (4.12), hence (4.11) and (3.12) are equivalent.

Comparing (4.13) and (4.9*b*) we get

$$\omega'_{13} = \omega_2. \tag{4.14}$$

Taking the Hodge star expression of equations (4.9*a*) and (4.14) respectively we get

$$\omega'_1 = \omega_1, \qquad \omega'_2 = \omega_{13}. \tag{4.15}$$

So after BT the new PSS $\boldsymbol{X}'$ satisfies

$$\begin{aligned} \mathrm{d}\boldsymbol{X}' &= \omega'_1\boldsymbol{e}'_1 + \omega'_2\boldsymbol{e}'_2 = \omega_1\boldsymbol{e}_1 + \omega_{13}(\cos\theta\boldsymbol{e}_2 + \sin\theta\boldsymbol{e}_3) \\ &= \mathrm{d}\boldsymbol{X} + \sin\theta\,\mathrm{d}\boldsymbol{e}_1. \end{aligned} \tag{4.16}$$

After integration we get

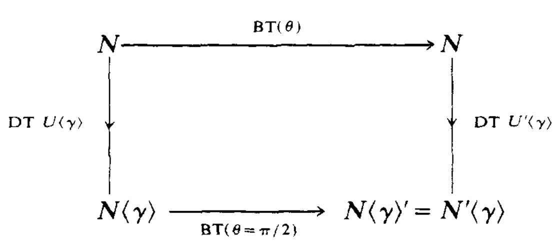
$$\boldsymbol{X}' = \boldsymbol{X}' + \sin\theta\boldsymbol{e}_1, \tag{4.17}$$

where the integral constant has been chosen to be zero.

Equation (4.6) shows that the corresponding normals of two PSS differ by a fixed angle θ around the common tangent axis $\boldsymbol{e}_1(x) = \boldsymbol{e}'_1(x)$. Simultaneously, equation (4.17) shows that the corresponding points of PSSs translate a fixed distance $\sin\theta$ along $\boldsymbol{e}_1(x)$, so that the common tangents between two PSSs constitute an equidistant congruence of lines. In short, we may see that under ansatz (3.14), the BT (4.1) of chiral field (with its consequence (4.4), (4.6)) realise the classical BT (4.7) between two PSSs.

5. BT and DT

Pohlmeyer (1976) has shown that the one parameter family of BT(θ) may be generated from the Bianchi transformation (BT(θ = π/2)) by DT as shown by the following diagram:



Here the BT are given by (4.10), (4.4) with the moving frames chosen as follows:

$$e_3(x)=N(x), \qquad e_3'(x)=N'(x),$$

$$e_3(x;\gamma)=N(x;\gamma)=N\langle\gamma\rangle, \qquad e_3'(x;\gamma)=N'(x;\gamma)=N'\langle\gamma\rangle;$$

$$\mathrm{i}e_1\cdot\sigma=\mathrm{i}e_1'\cdot\sigma=(1/\sin\theta)[N,N'], \tag{5.1}$$

$$\mathrm{i}e_1\langle\gamma\rangle\cdot\sigma=\mathrm{i}e_1\langle\gamma\rangle'\cdot\sigma=[N\langle\gamma\rangle,N\langle\gamma\rangle']. \tag{5.2}$$

While the DT are given by

$$N\langle\gamma\rangle=U^{-1}\langle\gamma\rangle NU\langle\gamma\rangle=u^{-1}\langle\gamma\rangle nu\langle\gamma\rangle,$$

$$N'\langle\gamma\rangle=U'^{-1}\langle\gamma\rangle N'U'\langle\gamma\rangle=u'^{-1}\langle\gamma\rangle nu'\langle\gamma\rangle, \tag{5.3}$$

where the dual transforming operators $U\langle\gamma\rangle$, $U'\langle\gamma\rangle$ satisfy

$$\partial_\mu U\langle\gamma\rangle=\tfrac{1}{2}(N\partial_\mu N(\cosh\varphi-1)+\varepsilon_{\mu\nu}N\partial^\nu N\sinh\varphi)U\langle\gamma\rangle,$$

$$\partial_\mu U'\langle\gamma\rangle=-(N'\partial_\mu N'(\cosh\varphi-1)+\varepsilon_{\mu\nu}N'\partial^\nu N'\sinh\varphi)U'\langle\gamma\rangle. \tag{5.4}$$

In this section we shall show that

$$N\langle\gamma\rangle'=N'\langle\gamma\rangle \tag{5.5}$$

implies a definite relation between the rotation angle θ and the dual scale γ as (5.11).

Using the RE in flat gauge one of us (Hou 1980) has already obtained this relation between θ and γ. Since in § 3 we have established the correspondence between chiral field and differential forms on PSS, we now reconsider this relation in reduced gauge in terms of differential forms.

In case of (5.5), we can further choose $e_1'\langle\gamma\rangle$ as

$$\mathrm{i}e_1'\langle\gamma\rangle\cdot\sigma=[N\langle\gamma\rangle,N'\langle\gamma\rangle]=\mathrm{i}e_1\langle\gamma\rangle'\cdot\sigma. \tag{5.6}$$

Then by using (3.22) for the DT$\langle\gamma\rangle$ and (4.9), (4.14), (4.15) for BT(θ) in 'common tangent coordinates', we get the following equalities:

Route 1 $X'^{\sim}\langle\gamma\rangle \xrightarrow{\mathrm{DT}\langle\gamma\rangle} X' \xrightarrow{\mathrm{BT}(\theta)} X$

$$\omega_{13}'^{\sim}=\cosh\varphi\omega_{13}'-\sinh\varphi\omega_2'=\cosh\varphi\omega_2+\sinh\varphi\omega_{13},$$

$$\omega_{23}'^{\sim}=\cosh\varphi\omega_{23}'+\sinh\varphi\omega_1'=\cosh\varphi\omega_{23}+\sinh\varphi\omega_1,$$

$$\omega_{12}'^{\sim}=\omega_{12}'=\cos\theta\omega_{12}+\sin\theta\omega_{13},$$

$$\omega_1'^{\sim}=\cosh\varphi\omega_1'+\sinh\varphi\omega_{23}'=\cosh\varphi\omega_1+\sinh\varphi\omega_{23},$$

$$\omega_2'^{\sim}=\cosh\varphi\omega_2'-\sinh\varphi\omega_{13}'=\cosh\varphi\omega_{13}-\sinh\varphi\omega_2. \tag{5.7}$$

Route 2 $\tilde{X}\langle\gamma\rangle' \xrightarrow{\mathrm{BT}(\theta=\pi/2)} \tilde{X}\langle\gamma\rangle \xrightarrow{\mathrm{DT}\langle\gamma\rangle} X$

$$\tilde{\omega}_{13}'=-\tilde{\omega}_{12}=\tilde{\omega}_2=-\omega_{12}$$

$$\tilde{\omega}_{23}'=\tilde{\omega}_{23}=\cosh\varphi\omega_{23}+\sinh\varphi\omega_1,$$

$$\tilde{\omega}_{12}'=\tilde{\omega}_{13}=\cosh\varphi\omega_{13}-\sinh\varphi\omega_2,$$

$$\tilde{\omega}_1'=\tilde{\omega}_1=\cosh\varphi\omega_1+\sinh\varphi\omega_{23},$$

$$\tilde{\omega}_2'=\tilde{\omega}_{13}=\cosh\varphi\omega_{13}-\sinh\varphi\omega_2. \tag{5.8}$$

The differential forms in (5.7) are required to be equal to those in (5.8) respectively

$$\tilde{\omega}'_{ij} = \omega'^{\sim}_{ij}, \qquad \tilde{\omega}'_{i} = \omega'^{\sim}_{i}. \tag{5.9}$$

Thus we get

$$\begin{aligned} -\omega_{12} &= \cosh \varphi\omega_2 - \sinh \varphi\omega_{13}, \\ \cos \theta\omega_{12} + \sin \theta\omega_{13} &= \cosh \varphi\omega_{13} - \sinh \varphi\omega_2. \end{aligned} \tag{5.10}$$

By comparing these equations with (4.13) we get finally

$$\cosh \varphi = 1/\sin \theta, \qquad \sinh \varphi = \cot \theta, \tag{5.11}$$

where

$$\cosh \varphi = (\gamma + \gamma^{-1})/2.$$

Conversely, if γ and θ satisfy (5.11), the differential forms obtained from two different routes are the same, hence the corresponding surfaces will be identical, and so yield the coincidence of their normals (5.5). In short the equality (5.5) will be fulfilled if and only if the relations (5.11) are satisfied.

Finally we express the post-BT operators $g'(x) = u'^{-1}(x; 1)$, $u'(x; \gamma)$, $U'(x; \gamma) = g'(x)u'(x; \gamma)$ in terms of the ante-BT operators $g(x)$, $u(x; \gamma)$, $U(x; \gamma)$ in the frame (5.2). Since geometrically, BT implies a rotation θ around the common tangent, we have

$$\begin{aligned} g'\sigma_i g'^{-1} = \boldsymbol{e}'_i \cdot \boldsymbol{\sigma} &= \exp(\mathrm{i}\boldsymbol{e}_1 \cdot \boldsymbol{\sigma}\theta/2)\boldsymbol{e}_i \cdot \boldsymbol{\sigma} \exp(-\mathrm{i}\boldsymbol{e}_1 \cdot \boldsymbol{\sigma}\theta/2) \\ &= \exp(\mathrm{i}\boldsymbol{e}_1 \cdot \boldsymbol{\sigma}\theta/2) g\sigma_i g^{-1} \exp(-\mathrm{i}\boldsymbol{e}_1 \cdot \sigma\theta/2), \end{aligned}$$

then by virtue of the Schur lemma

$$g'(x) = \exp(\mathrm{i}\boldsymbol{e}_1(x) \cdot \boldsymbol{\sigma}\theta/2) g(x) = g(x) \exp(\mathrm{i}\sigma_1 \theta/2). \tag{5.12}$$

Similarly we obtain

$$\begin{aligned} u'(x; \gamma) &= u(x; \gamma) \exp(-\mathrm{i}\boldsymbol{e}_1(x; \gamma) \cdot \boldsymbol{\sigma}\pi/4) \\ &= \exp(-\mathrm{i}\sigma_1 \pi/4) u(x; \gamma); \end{aligned} \tag{5.13}$$

$$\begin{aligned} U'(x; \gamma) &= \exp(\mathrm{i}\boldsymbol{e}_1(x) \cdot \boldsymbol{\sigma}\theta/2) U(x; \gamma) \exp(-\mathrm{i}\boldsymbol{e}_1(x; \gamma) \cdot \boldsymbol{\sigma}\pi/4) \\ &= g \exp(\mathrm{i}\sigma_1(\theta - \pi/2)/2) u(x; \gamma). \end{aligned} \tag{5.14}$$

6. RES and its relation with the AKNS systems

In § 4, we assume that the BT equations (4.1) are fulfilled. Then if $N(x)$ is a solution of the ELES (2.4), $N'(x)$ will also be a solution of (2.4). But we must make certain whether or not the BT equations (4.1) are consistent for any given $N(x)$. In order to solve this problem we introduce the matrix B

$$N' = NB. \tag{6.1}$$

Substituting this into (4.1), after eliminating $N'(x)$, we get the equations about B. If these equations are consistent, we can find out B, then from (6.1) N' will be obtained. In this section we shall show that these equations are just the RES, and their integrability conditions are the ELE and GCES. We also discover that (4.1) is a strong BT.

Thus from (4.6), (5.10) we get

$$B+B^{+}=2\tanh\varphi I. \tag{6.2}$$

Let

$$\hat{B}=\cosh\varphi(B-B^{+})/2. \tag{6.3}$$

Combine it with (4.1), we get the matrix RE.

$$\mathscr{D}_{\mu}\hat{B}\equiv\partial_{\mu}\hat{B}+[H_{\mu},\hat{B}]=-\varepsilon_{\mu\nu}(\tilde{K}^{\nu}+\hat{B}\tilde{K}^{\nu}\hat{B}), \tag{6.4}$$

From $\mathscr{D}_{\mu}\tilde{K}^{\mu}=0$ and (2.33) we see easily that the above equations are integrable. Substitute the obtained $\hat{B}$ in (6.3), (6.2) to get B, then (6.1) gives the new solution $N'(x)$.

The operator $B(x)$ expresses a rotation from $N(x)$ to $N'(x)$, i.e. rotates a certain angle θ about the common tangent of two PSSs. Therefore obtaining $B(x)$ is equivalent to finding the direction of the common tangent. Let δ be the angle between the required direction of the common tangent and the first axis $\boldsymbol{e}_1(x)$ of a certain moving frame (in § 4 we have adopted the particular case $\delta=0$).

$$\begin{aligned}B=NN'&=\cos\theta I+\mathrm{i}\sin\theta\boldsymbol{\sigma}\cdot(\boldsymbol{e}_1\cos\delta+\boldsymbol{e}_2\sin\delta)\\&=\exp(\theta\hat{B}),\end{aligned} \tag{6.5}$$

where

$$\hat{B}=\mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{e}_1\cos\delta+\boldsymbol{e}_2\sin\delta). \tag{6.6}$$

This satisfies

$$\hat{B}^2=-I,\qquad \hat{B}^{+}=-\hat{B}, \tag{6.7}$$

and can be expressed as

$$\hat{B}=\mathrm{i}(2P-I), \tag{6.8}$$

where P is the projection operator

$$P^2=P. \tag{6.9}$$

Substituting (6.8) in (6.4) yields

$$\begin{aligned}&P(\mathscr{D}_{\mu}+\tilde{K}_{\mu}\langle-\mathrm{i}\gamma\rangle)(P-I)=0,\\&(I-P)(\mathscr{D}_{\mu}+\tilde{K}_{\mu}\langle\mathrm{i}\gamma\rangle)P=0.\end{aligned} \tag{6.10}$$

Thus the matrix RE is equivalent to the equations of projection operator P in the Riemann–Hilbert formulation of Zakharov and Mikhailov (1978).

Now we simplify the matrix equations into scalar equations by using the reduced gauge, then (6.4) becomes

$$\mathrm{d}\hat{b}+[h_{\mu}\,\mathrm{d}x^{\mu},\hat{b}]=[{}^{*}\tilde{k}_{\mu}\,\mathrm{d}x^{\mu},\hat{b}]\hat{b}, \tag{6.11}$$

where

$$\begin{aligned}\hat{b}&=u\langle\gamma\rangle\hat{B}u^{-1}\langle\gamma\rangle=\mathrm{i}(\cos\delta\sigma_1+\sin\delta\sigma_2)\\&=\mathrm{i}\begin{pmatrix}0&\Gamma\\\Gamma^{-1}&0\end{pmatrix},\end{aligned} \tag{6.12}$$

$$\Gamma=\exp(-\mathrm{i}\delta),\qquad \Gamma^{-1}=\exp(\mathrm{i}\delta). \tag{6.13}$$

Substituting (6.12) and (6.13) in (6.11), we actually get two scalar RES:

$$d\Gamma = -(\tilde{\omega}_1 - i\tilde{\omega}_2)/2 + i\omega_{12}\Gamma + (\tilde{\omega}_1 + i\tilde{\omega}_2)\Gamma^2/2, \tag{6.14}$$

$$d\Gamma^{-1} = -(\tilde{\omega}_1 + i\tilde{\omega}_2)/2 - i\omega_{12}\Gamma^{-1} + (\tilde{\omega}_1 - i\tilde{\omega}_2)/2. \tag{6.15}$$

Both these RES have su(1, 1) symmetry. They are nothing other than the nonlinear representation of the following linear scattering equations

$$d\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} i\omega_{12} & \tilde{\omega}_1 - i\tilde{\omega}_2 \\ \tilde{\omega}_1 + i\tilde{\omega}_2 & -i\omega_{12} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{6.16}$$

where $\binom{v_1}{v_2}$ is a column of the matrix $v\langle\gamma\rangle$ of equation (3.25). As can be easily seen, the scalars

$$\Gamma = -v_1/v_2, \qquad \Gamma^{-1} = -v_2/v_1, \tag{6.17}$$

satisfy equations (6.14), (6.15) respectively. So the RES of the chiral field are connected with the RES of the linear scattering equation in the AKNS systems, and both are related to the first fundamental form $\Omega^1\langle\gamma\rangle$ of the surface with negative constant curvature. Its geometrical implication is obvious by noting that the solution $\Gamma = \exp(-i\delta)$ determines the rotation from the axis $\boldsymbol{e}_1$ of the chosen frames to the common tangent in the BT. Finally, we show briefly the origin of the ordinary BT in the sine-Gordon equation. From § 4 we know that $\omega_2 = \omega'_{13}$. Writing this equation explicitly in terms of the expressions in table 1, we get $\alpha = \beta'$, $\alpha' = \beta$, where $\alpha(\alpha')$ are the angles between asymptotes ante-(post-)BT while $\beta(\beta')$ are the angles between the curvature line and the common tangent. Now in ordinary iso-spectrum formulation one of the axes has been chosen along an asymptote, hence the angle $-\delta$ equals $(\alpha - \beta)/2 = (\alpha - \alpha')/2$. Thus we get the geometrical explanation of this ansatz, which is just the keypoint for getting the BT from the RES.

Table 1. The fundamental differential forms on PSS.

In principal curvature coordinates	In asymptotic coordinates	In common tangent coordinates
$\omega_1 = \cos\frac{1}{2}\alpha(d\xi + d\eta)$	$\omega_1 = d\xi + \cos\alpha\, d\eta$	$\omega_1 = \cos\frac{1}{2}(\alpha-\beta)\,d\xi + \cos\frac{1}{2}(\alpha+\beta)\,d\eta$
$\omega_2 = \sin\frac{1}{2}\alpha(d\xi - d\eta)$	$\omega_2 = -\sin\alpha\, d\eta$	$\omega_2 = \sin\frac{1}{2}(\alpha-\beta)\,d\xi - \sin\frac{1}{2}(\alpha-\beta)\,d\eta$
$\omega_{23} = \cos\frac{1}{2}\alpha(d\xi - d\eta)$	$\omega_{23} = d\xi - \cos\alpha\, d\eta$	$\omega_{23} = \cos\frac{1}{2}(\alpha-\beta)\,d\xi - \cos\frac{1}{2}(\alpha+\beta)\,d\eta$
$\omega_3 = -\sin\frac{1}{2}\alpha(d\xi + d\eta)$	$\omega_{13} = -\sin\alpha\, d\eta$	$\omega_{13} = -\sin\frac{1}{2}(\alpha-\beta)\,d\xi - \sin\frac{1}{2}(\alpha+\beta)\,d\eta$
$\omega_{12} = \frac{1}{2}(\alpha_\xi\, d\xi - \alpha_\eta\, d\eta)$	$\omega_{12} = \alpha_\xi\, d\xi$	$\omega_{12} = \frac{1}{2}(\alpha_\xi + \beta_\xi)\,d\xi - \frac{1}{2}(\alpha_\eta - \beta_\eta)\,d\eta$

7. Nöether current of the BT and infinitely many more conservation laws

In this section we introduce an infinitesimal BT with parameter γ, which shifts the Lagrangian of the chiral model by a total divergence. Therefore it generates a Nöether current with parameter γ. Then in reduced gauge, we express this Nöether current by quantities on PSS, which are equivalent to ordinary local current in the AKNS systems.

Let

$$\delta N = N\hat{B}\varepsilon, \tag{7.1}$$

where $\hat{B}$ is a solution of (6.4). (For simplicity we omit the constant ε in future.)

Defining

$$j_\mu(x;\gamma) = \mathrm{Tr}\left(\frac{\delta L}{\delta\partial_\mu N}\delta N\right) = \mathrm{Tr}(K_\mu \hat{B}), \tag{7.2}$$

after using the equation of motion (2.7), we have

$$\partial_\mu j^\mu = \delta L = \mathrm{Tr}(K_\mu \partial^\mu \hat{B}). \tag{7.3}$$

Thus if $\mathrm{Tr}(K_\mu \partial^\mu \hat{B}) = 0$ as in the case of $\gamma = 1$, then j_μ (7.3) is the ordinary conserved Nöether current. Generally $\mathrm{Tr}(K_\mu \partial^\mu \hat{B})$ is not zero, but we will show that it can be expressed as a total divergence

$$\mathrm{Tr}(K_\mu \partial^\mu \hat{B}) = -\partial_\mu i^\mu(x;\gamma), \tag{7.4}$$

such that we may construct a generalised conserved Nöether current

$$J_\mu(x;\gamma) = j_\mu(x;\gamma) + i_\mu(x;\gamma). \tag{7.5}$$

By use of the RE (6.4) and equation (2.42), it is easy to show

$$\begin{aligned}\mathrm{Tr}(K_\mu \partial^\mu \hat{B}) &= -2\sinh\varphi\, \mathrm{Tr}(K_\mu K^\mu + K_\mu \hat{B} K^\mu \hat{B}) \\ &= -\tanh\varphi\partial_\mu\, \mathrm{Tr}({}^*K^\mu \hat{B}) = -\partial_\mu i^\mu(x;\gamma).\end{aligned} \tag{7.6}$$

Finally we get the conserved current

$$\begin{aligned}J_\mu(x;\gamma) &= j_\mu(x;\gamma) + i_\mu(x;\gamma) = \mathrm{sech}\,\varphi\, \mathrm{Tr}(\tilde{K}_\mu \hat{B}) \\ &= \mathrm{sech}\,\varphi\, \mathrm{Tr}(\tilde{K}_\mu \hat{B}),\end{aligned} \tag{7.7}$$

where the gauge covariance has been used. More simply, utilising the fact that in common tangent coordinates $\Gamma = 1$, then from (6.17), (3.26) it is easy to get

$${}^*J_\mu\, \mathrm{d}x^\mu = \sin\theta\tilde{\omega}_1. \tag{7.8}$$

Taking the exterior differentiation of the above equation we get

$$\sin\theta\, \mathrm{d}\tilde{\omega}_1 = (\partial^* J_\mu/\partial x^\nu)\varepsilon^{\mu\nu}\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 = -(\partial J^\mu/\partial x^\mu)\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 = 0.$$

So the conservation of current can be expressed as

$$\mathrm{d}\tilde{\omega}_1 = 0. \tag{7.9}$$

Since in common tangent coordinates $\omega_{12} = -\tilde{\omega}_2$, equation (7.9) is also a consequence of equation (3.16). The $J_\mu(x;\gamma)$ constitutes a one-parameter family of conserved currents, hence they can be expanded in inverse powers of γ to get the infinite number of conservation laws, which is the same result as that obtained by expansion of Riccati functions.

8. Discussion

The results of this paper may be generalised into the $O(N)$ σ-model, $CP(N)$ model etc by using localised involution operator $N(x)$ ($[N, \eta)] = 0$, $\{N, \kappa\} = 0$, $\kappa + \eta = g$).

Then we can show the existence of a corresponding two-dimensional surface X with $\mathrm{d}X \simeq [N, {}^*\mathrm{d}N]$. Its first and second fundamental forms are mutually dual, the asymptotes go along lightlike derivatives. Its Gauss image induces the chiral solution but the curvature of these surfaces is not generally constant. But even in the reduced gauge, the RE remains as a matrix equation in some subgroup. Details are under investigation.

Our PSSS are a concrete example of the 'soliton surface' called by Sym (1982). For the Ernst (1968) equation written in the Maison (1979) normalised form we can obtain a similar surface with 'lightlike' asymptotes. However its curvature instead of being constant, now becomes proportional to $|g_{\mu\nu}|^{-1}$. In this case the BT may also be expressed explicitly as the transformation between two corresponding surfaces, we have already found expressions similar to (4.1), (4.17), (6.4) and (6.10). These are consequences of the Lelieuvre formulae (Eisenhart 1909).

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Appendix. Relation between the equation of the chiral field and the sine-Gordon equation

From the equation of motion we get conservation of energy and momentum as

$$\partial_\eta \operatorname{Tr}(N_\xi^2) = 0 \qquad (N_\xi \equiv \partial_\xi N),$$
$$\partial_\xi \operatorname{Tr}(N_\eta^2) = 0 \qquad (N_\eta \equiv \partial_\eta N). \tag{A1}$$

So we may assume

$$\operatorname{Tr}(N_\xi N_\eta)/2 = \boldsymbol{N}_\xi \cdot \boldsymbol{N}_\eta = -r(\xi)s(\eta)\cos\alpha(\xi, \eta). \tag{A2}$$

Choosing

$$\boldsymbol{e}_1 = (r^{-1}\boldsymbol{N}_\xi + s^{-1}\boldsymbol{N}_\eta)/2\sin\alpha/2,$$
$$\boldsymbol{e}_2 = -(r^{-1}\boldsymbol{N}_\xi - s^{-1}\boldsymbol{N}_\eta)/2\cos\alpha/2,$$
$$\boldsymbol{e}_3 = \boldsymbol{N}, \tag{A3}$$

gives

$$\mathrm{d}\boldsymbol{e}_3 = (r\,\mathrm{d}\xi + s\,\mathrm{d}\eta)\sin\tfrac{1}{2}\alpha\boldsymbol{e}_1 - (r\,\mathrm{d}\xi - s\,\mathrm{d}\eta)\cos\tfrac{1}{2}\alpha\boldsymbol{e}_2. \tag{A4}$$

By comparing with the Weingarten formula we get

$$\omega_{13} = -(r\,\mathrm{d}\xi + s\,\mathrm{d}\eta)\sin\tfrac{1}{2}\alpha,$$
$$\omega_{23} = (r\,\mathrm{d}\xi - s\,\mathrm{d}\eta)\cos\tfrac{1}{2}\alpha. \tag{A5}$$

By applying the GCEs we obtain

$$\alpha_{\xi\eta} = r(\xi)s(\eta)\sin\alpha, \tag{A6}$$

$$\omega_{12} = \tfrac{1}{2}(\alpha_\xi \, d\xi - \alpha_\eta \, d\eta). \tag{A7}$$

The equation (A6) may be called the generalised sine-Gordon equation. By further use of the dual symmetry of the chiral field we get

$$\omega_1 = {}^*\omega_{23} = (r\, d\xi + s\, d\eta)\cos\tfrac{1}{2}\alpha,$$

$$\omega_2 = -{}^*\omega_{13} = (r\, d\xi - s\, d\eta)\sin\tfrac{1}{2}\alpha. \tag{A8}$$

From (A7) and (A8) it is easy to show that the corresponding surface has Gauss curvature $K = -1$.

Since the chiral fields are conformal invariant, corresponding to any chiral field there exists a solution with

$$N'(\xi', \eta') = N(\xi(\xi'), \eta(\eta')),$$

where $\xi(\xi')$, $\eta(\eta')$ are solutions of

$$d\xi' = r(\xi)\, d\xi, \qquad d\eta' = s(\eta)\, d\eta, \tag{A9}$$

then

$$\mathrm{Tr}(N_{\xi'}^2)/2 = \mathrm{Tr}(N_{\eta'}^2)/2 = 1, \qquad \mathrm{Tr}(N_{\xi'}N_{\eta'})/2 = -\cos\alpha(\xi, \eta). \tag{A10}$$

The forms on PSS become

$$\omega_{12} = (\alpha_{\xi'}\, d\xi' - \alpha_{\eta'}\, d\eta')/2$$

$$\omega_1 = \cos\tfrac{1}{2}\alpha(d\xi' + d\eta') = \cos\tfrac{1}{2}\alpha\, dt',$$

$$\omega_2 = \sin\tfrac{1}{2}\alpha(d\xi' - d\eta') = \sin\tfrac{1}{2}\alpha\, dx'. \tag{A11}$$

Their integrability conditions turn out to be the sine-Gordon equation

$$\alpha_{\xi'\eta'} = \sin\alpha. \tag{A12}$$

The solution, which satisfies the normalisation condition (A10) will be called the normal chiral field. We stress that although under conformal transformation of independent variables

$$\xi = \xi(\xi'), \qquad \eta = \eta(\eta'), \tag{A13}$$

a different solution (conformally similar) of the chiral field equation has been obtained:

$$N'(\xi', \eta') = N(\xi(\xi'), \eta(\eta')),$$

$$[N_{\xi\eta}, N] = 0, \qquad [N'_{\xi'\eta'}, N'] = 0. \tag{A14}$$

However, since

$$N'_\xi\, d\xi' = N_\xi\, d\xi, \qquad K'_{\xi'}\, d\xi' = K_\xi\, d\xi, \qquad {}^*K_\xi\, d\xi' = {}^*K_\xi\, d\xi \text{ etc}$$

the fundamental forms are the same, which correspond to one and the same PSS. However, ∂_ξ, ∂_η (or $\partial_{\xi'}$, $\partial_{\eta'}$) remain the asymptotic directions. Only the scale of the image from the base space ξ, η to PSS has been changed. We must distinguish the conformal transformation of independent variables from the dual expansion and contraction. The dual expansion and contraction is that, which with respect to any

Table 2. The expressions of normal chiral field under the explicit reduced gauge in terms of quantities of PSS.

Algebra and fundamental forms	Chiral field	In principal curvature coordinates	In asymptotic coordinates	In common tangent coordinates
su(2)	$-h_\xi-\gamma^{-1}k_\xi$	$\frac{1}{2}\mathrm{i}\begin{pmatrix}\alpha_\xi/2 & \gamma^{-1}\mathrm{e}^{-\mathrm{i}\alpha/2}\\ \gamma^{-1}\mathrm{e}^{\mathrm{i}\alpha/2} & -\alpha_\xi/2\end{pmatrix}$	$\frac{1}{2}\mathrm{i}\begin{pmatrix}\alpha_\xi & \gamma^{-1}\\ \gamma^{-1} & -\alpha_\xi\end{pmatrix}$	$\frac{1}{2}\mathrm{i}\begin{pmatrix}(\alpha_\xi+\beta_\xi)/2 & \gamma^{-1}\mathrm{e}^{-\mathrm{i}(\alpha-\beta)/2}\\ -\gamma^{-1}\mathrm{e}^{\mathrm{i}(\alpha-\beta)/2} & -(\alpha_\xi+\beta_\xi)/2\end{pmatrix}$
$\Omega^{\mathrm{II}}=\frac{1}{2}\begin{pmatrix}\mathrm{i}\omega_{12} & -\omega_{13}+\mathrm{i}\omega_{23}\\ \omega_{13}+\mathrm{i}\omega_{23} & -\mathrm{i}\omega_{12}\end{pmatrix}$	$-h_\eta-\gamma k_\eta$	$\frac{1}{2}\mathrm{i}\begin{pmatrix}-\alpha_\eta/2 & -\gamma\,\mathrm{e}^{\mathrm{i}\alpha/2}\\ -\gamma\,\mathrm{e}^{-\mathrm{i}\alpha/2} & \alpha_\eta/2\end{pmatrix}$	$\frac{1}{2}\mathrm{i}\begin{pmatrix}0 & -\gamma\,\mathrm{e}^{\mathrm{i}\alpha}\\ -\gamma\,\mathrm{e}^{\mathrm{i}\alpha} & 0\end{pmatrix}$	$\frac{1}{2}\mathrm{i}\begin{pmatrix}-(\alpha_\eta-\beta_\eta)/2 & -\gamma\,\mathrm{e}^{\mathrm{i}(\alpha+\beta)/2}\\ -\gamma\,\mathrm{e}^{-\mathrm{i}(\alpha+\beta)/2} & (\alpha_\eta-\beta_\eta)/2\end{pmatrix}$
su(1, 1)	$-h_\xi-\mathrm{i}\gamma^{-1*}k_\xi$	$\frac{1}{2}\begin{pmatrix}\mathrm{i}\alpha_\xi/2 & \gamma^{-1}\mathrm{e}^{-\mathrm{i}\alpha/2}\\ \gamma^{-1}\mathrm{e}^{\mathrm{i}\alpha/2} & -\mathrm{i}\alpha_\xi/2\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}\mathrm{i}\alpha_\xi & \gamma^{-1}\\ \gamma^{-1} & -\mathrm{i}\alpha_\xi\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}\mathrm{i}(\alpha_\xi+\beta_\xi)/2 & \gamma^{-1}\mathrm{e}^{-\mathrm{i}(\alpha-\beta)/2}\\ \gamma^{-1}\mathrm{e}^{\mathrm{i}(\alpha-\beta)/2} & -\mathrm{i}(\alpha_\xi+\beta_\xi)/2\end{pmatrix}$
$\Omega^{\mathrm{I}}=\frac{1}{2}\begin{pmatrix}\mathrm{i}\omega_{12} & \omega_1-\mathrm{i}\omega_2\\ \omega_1+\mathrm{i}\omega_2 & -\mathrm{i}\omega_{12}\end{pmatrix}$	$-h_\eta-\mathrm{i}\gamma^{*}k_\eta$	$\frac{1}{2}\begin{pmatrix}-\mathrm{i}\alpha_\eta/2 & \gamma\,\mathrm{e}^{\mathrm{i}\alpha/2}\\ \gamma\,\mathrm{e}^{-\mathrm{i}\alpha/2} & \mathrm{i}\alpha_\eta/2\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}0 & \gamma\,\mathrm{e}^{\mathrm{i}\alpha}\\ \gamma\,\mathrm{e}^{-\mathrm{i}\alpha} & 0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}-\mathrm{i}(\alpha_\eta-\beta_\eta)/2 & \gamma\,\mathrm{e}^{\mathrm{i}(\alpha+\beta)/2}\\ \gamma\,\mathrm{e}^{-\mathrm{i}(\alpha+\beta)/2} & \mathrm{i}(\alpha_\eta-\beta_\eta)/2\end{pmatrix}$
sl(2r)	$-h_\xi-\mathrm{i}\gamma^{-1*}k_\xi$	$\frac{1}{2}\begin{pmatrix}\gamma^{-1}\cos\frac{1}{2}\alpha & \gamma^{-1}\sin\frac{1}{2}\alpha+\frac{1}{2}\alpha_\xi\\ \gamma^{-1}\sin\frac{1}{2}\alpha-\frac{1}{2}\alpha_\xi & -\gamma^{-1}\cos\frac{1}{2}\alpha\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}\gamma^{-1} & \alpha_\xi\\ -\alpha_\xi & -\gamma^{-1}\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}\gamma^{-1}\cos\frac{1}{2}(\alpha-\beta) & \gamma^{-1}\sin\frac{1}{2}(\alpha-\beta)+\frac{1}{2}(\alpha_\xi+\beta_\xi)\\ \gamma^{-1}\sin\frac{1}{2}(\alpha-\rho)-\frac{1}{2}(\alpha_\xi+\beta_\xi) & -\gamma^{-1}\cos\frac{1}{2}(\alpha-\beta)\end{pmatrix}$
$\Omega^{\mathrm{I}}=\frac{1}{2}\begin{pmatrix}\omega_1 & \omega_2+\omega_{12}\\ \omega_2-\omega_{12} & -\omega_1\end{pmatrix}$	$-h_\eta-\mathrm{i}\gamma^{*}k_\eta$	$\frac{1}{2}\begin{pmatrix}\gamma\cos\frac{1}{2}\alpha & -\gamma\sin\frac{1}{2}\alpha-\frac{1}{2}\alpha_\eta\\ -\gamma\sin\frac{1}{2}\alpha+\frac{1}{2}\alpha_\eta & -\gamma\cos\frac{1}{2}\alpha\end{pmatrix}$	$\frac{1}{2}\gamma\begin{pmatrix}\cos\alpha & -\sin\alpha\\ -\sin\alpha & -\cos\alpha\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}\gamma\cos\frac{1}{2}(\alpha+\beta) & -\gamma\sin\frac{1}{2}(\alpha+\beta)-\frac{1}{2}(\alpha_\eta-\beta_\eta)\\ -\gamma\sin\frac{1}{2}(\alpha+\beta)+\frac{1}{2}(\alpha_\eta-\beta_\eta) & -\gamma\cos\frac{1}{2}(\alpha+\beta)\end{pmatrix}$

given solution $N(\xi, \eta)$, enables one to find a new solution $N(\xi, \eta; \gamma)$ such that

$$|N_\xi(\xi, \eta; \gamma)| = \gamma^{-1}|N_\xi(\xi, \eta)|, \qquad |N_\eta(\xi, \eta; \gamma)| = \gamma|N_\eta(\xi, \eta)|,$$

and

$$\boldsymbol{N}_\xi(\xi, \eta; \gamma)\cdot\boldsymbol{N}_\eta(\xi, \eta; \gamma) = \boldsymbol{N}_\xi(\xi, \eta)\cdot\boldsymbol{N}_\eta(\xi, \eta),$$

where all equalities stand on the same ξ, η. But in the conformal transformation (A14) ξ', η' and ξ, η are different points. It is easy to see that after dual expansion and contraction all the first and second fundamental forms have been changed, we arrive at a new PSS with Gauss image $N(\xi, \eta; \gamma)$; meanwhile the parameter γ of expansion and contraction corresponds to the ordinary spectrum parameter.

On PSS we can choose the moving frame as (A3), which may be called the principal curvature coordinates. If one of the asymptotic directions has been chosen as $\boldsymbol{e}_1(x)$ (or $\boldsymbol{e}_2(x)$), they may be called asymptotic coordinates. In these coordinates the spectrum parameter γ may be expressed explicitly as the eigenvalue of the linear scattering equation. To consider the BT as in §§ 4 and 5, it is convenient to choose the common tangent as $\boldsymbol{e}_1(x)$, which have been called the common tangent coordinates. In table 1 we show the fundamental differential forms of PSS in various coordinates. In table 2 we give the explicit expressions of the normal chiral field under the reduced gauge in terms of quantities of PSS.

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